HORSESHOE PERIODIC ORBITS FOR SATURN COORBITAL SATELLITES

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Abstract

We consider the motion of coorbital satellites in the framework of the Restricted Three-Body problem for $\mu = 6.5 \cdot 10^{-9}$. We show a mechanism to generate horsehoe periodic orbits and we compute some families of horseshoe periodic orbits when varying the Jacobian constant. We conclude that there exist stable horseshoe periodic orbits which fit with the motion of Saturn coorbital satellites Janus and Epimetheus.

Key words and expressions: coorbital motion - horseshoe periodic orbits - Restricted problem.

1. Introduction

In 1981 the successful Voyager flights to Saturn confirmed the existence of two small satellites of Saturn, Janus (1908S1) and Epimetheus (1980S3), and provided an estimate of their masses as well as their orbital elements. These coorbital satellites turned out to be librating in horseshoe orbits, in a convenient rotating system, since their semimajor axis are only 50 km apart, they can approach within 15.000 km, but when they are close to each other, their mutual gravitational interaction prevents a collision and switches their orbits.

Several authors have dealt with the coorbital motion in the framework of the planar three-body problem: from an astronomical point of view (see Taylor 1981, Dermott and co-workers 1981, and Yoder et al. 1983), and also mathematical theories have been developed (see Spirig and Waldvogel 1985 and 1988 and Hénon and Petit 1986).

Our approach considers the *planar circular restricted three-body problem*, where the primaries are Saturn and Janus, that is, with a mass parameter $\mu = 6.5 \cdot 10^{-9}$, and Epimetheus (the smaller satellite) is the third body of infinitessimal mass.

In this communication, we show the existence of new families of horseshoe periodic orbits which fit the actual coorbital motion in this *simple* model. We also present a mechanism of generation of horseshoe periodic orbits as a transition from the $\mu = 0$ case to the $\mu > 0$ and small one. This mechanism gives answer to the natural question about the origin and location of these horseshoe periodic orbits. We carry out a numerical exploration for this particular value of μ, we will compute some families of horseshoe periodic orbits in a *systematic* way, and we will look for stable ones which fit the real motion of 1980S1 and 1980S3.

2. The restricted three-body problem

We consider a system of three bodies in an inertial (also called sidereal) reference system: two bodies (called primaries) of masses $1-\mu$ and μ (in suitable units), describing circular orbits about their common center of mass (the origin of coordinates) in a plane, and a particle of infinitesimal mass which moves in the same plane under the gravitational effect of the primaries but has negligible effect on their motion. The problem of describing the motion of the particle is the planar circular restricted three-body problem (RTBP). The equations of motion in a rotating (also called synodical) system of coordinates, x and y, which rotates with the primaries are, in suitable units (see Szebehely, 1967)

$$x'' - 2y' = \frac{\partial \Omega}{\partial x}, \tag{1}$$

$$y'' + 2x' = \frac{\partial \Omega}{\partial y}, \tag{2}$$

where

$$\Omega(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$$

$r_1^2 = (x-\mu)^2 + y^2$, $r_2^2 = (x-\mu+1)^2 + y^2$ are the distances between the particle and the bigger and smaller primaries respectively, and $'$ stands for d/dt. It is well known that these equations have the so called Jacobi first integral

$$x'^2 + y'^2 = 2\Omega(x,y) - C \tag{3}$$

We also recall that the RTBP has 5 equilibrium points: the collinear points, L_1, L_2 and L_3, and the equilateral ones, L_4 and L_5. If the value of the Jacobi constant at the equilibrium point is computed, $C_i = C(L_i)$, then for any value of $\mu \in (0, 1/2)$,

$$3 = C_4 = C_5 < C_3 < C_1 < C_2,$$

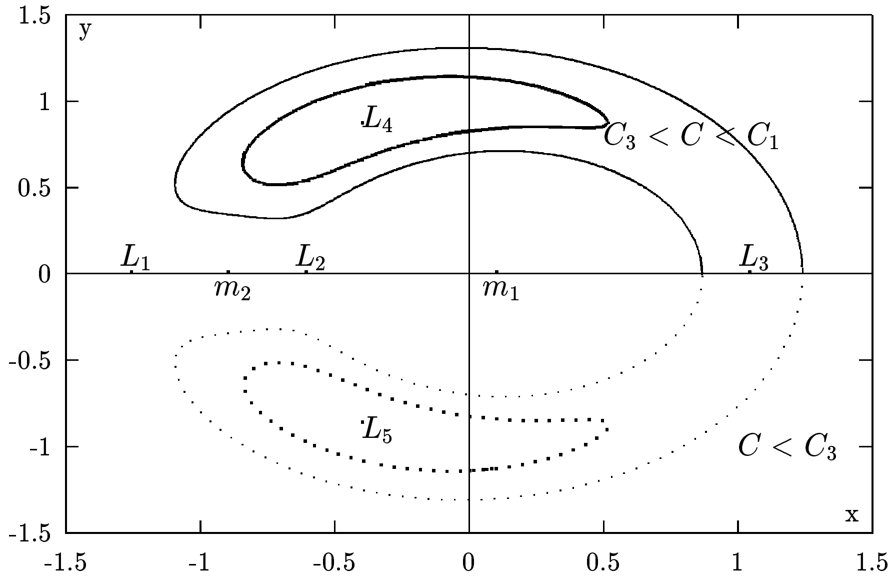

Figure 1.—Zero velocity curves for $\mu > 0$. .

and $C_3 = C_1$ for $\mu = 1/2$.

On one hand, we want to know the suitable region in the plane (x,y) to look for horseshoe periodic orbits. To do so, we just recall that for any value of μ, one has the possible regions of motion (whose boundaries are called zero velocity curves) according to the value of the Jacobi constant C (see more details in Szebehely 1967, Chapter 4). An insight of those possible regions of motion (see Figure 1) shows that horseshoe periodic orbits may take place only for $C < C_1$ and for values of x and y close to the zero velocity curves (we will precise the meaning of close later on).

On the other hand, for a fixed $C < C_1$ we may expect to have *many* horseshoe periodic orbits for $\mu > 0$ and very small. The reason remains in the RTBP for $\mu = 0$, as we shall see in the next section.

We briefly recall the RTBP with the particular value of the mass parameter $\mu = 0$. In this case, any point with $x' = y' = 0$ on the circle S^1 (centered at the origin and of radius one) is an equilibrium point. The forbidden regions of motion are: for $3 < C$, the interior of a ring of radius $r_i < 1$ and $r_o > 1$ which are solutions of the equation $r^3 - Cr + 2 = 0$, with $r > 0$; the plane except the circle S^1, for $C = 3$, and the whole plane for $C < 3$ (see Figure 2).

3. Mechanism of generation of horseshoe periodic orbits. From $\mu = 0$ to $\mu > 0$

It is well known (see Szebehely, 1967) that, for $\mu = 0$, any periodic orbit of the RTBP in synodical coordinates comes from rotating a circular orbit or a particular elliptical one

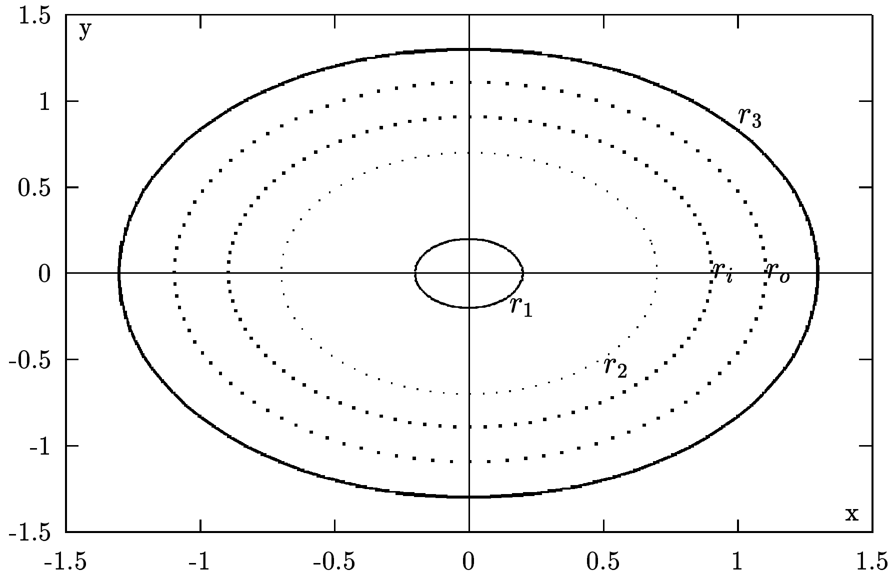

Figure 2.—$\mu = 0$. For $C > 3$, the motion is possible inside the disk of radius $r_i < 1$ and outside the disk of radius $r_o > 1$. For $C = 3$, the ring becomes the circle of radius $r_i = r_o = 1$. For $C < 3$ the motion is possible in the whole plane.

in the sidereal (inertial) frame. More concretely, fixed a value of C the possible sidereal ellipses (that is, in sidereal cordinates), with semimajor axis a, semiminor axis b and eccentricity e are given by the relation

$$\pm b = \pm a\sqrt{1-e^2} = \frac{C\sqrt{a}}{2} - \frac{1}{2\sqrt{a}} \tag{4}$$

(see Figure 3) and only those with $a = (p/q)^{2/3}$, for some $p, q \in N$, that is with rational mean motion, give rise to periodic orbits in synodical coordinates; of course, a and b are always positive but the signs $+$, $-$ assign a sense to describe the motion: (+), the corresponding sidereal orbit is direct, and (-), the sidereal orbit is retrograde.

As far as synodical circular orbits are concerned, we expect that for $\mu > 0$ and very small, we *may* have a horseshoe periodic orbit with the two perpendicular crossings $x_0 = x(t=0) > 0$ and $x_1 = x(t = T/2) > 0$ *very* close to the values r_2 and r_3. According to the singular perturbation theory carried out by Spirig and Waldvogel, these two circular orbits, outside a neighborhood of the small primary, would become the outer approximation of the horseshoe periodic orbit. It is also clear that the effect of the small primary with mass $\mu > 0$ may cause the typical shape of 'return' close to the primary. As we shall see in the next section, this is precisely the case. So, we may say that the two circular orbits, of radius $r_2 < 1 < r_3$, and r_2, r_3 close to 1, generate *one* horseshoe periodic orbit.

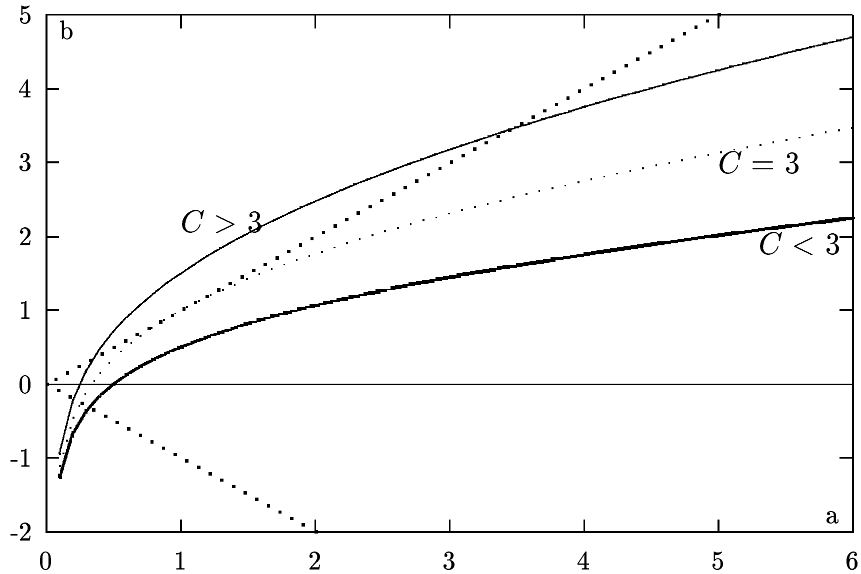

Figure 3.—For fixed C, any value of $a = (p/q)^{2/3}$, with $p, q \in N$ and b in the corresponding curve are the semiaxes of a sidereal ellipse which becomes a periodic orbit in synodical coordinates.

The same reasoning applies to synodical periodic orbits obtained from rotating sidereal ellipses with rational mean motion. However, as we have seen in Figure 3, they exist for *any* value of C, so for a fixed value of C close to 3 (greater than, equal to, or smaller than 3), we have a dense set of synodical periodic orbits , so it is likely that many pairs of them are the natural candidates to provide horseshoe periodic orbits (outside a neighborhood of the small primary) for $\mu > 0$ and small. In summary, we have given a natural mechanism to generate horseshoe periodic orbits, and we can expect for a fixed value of C close to 3, and $\mu > 0$ small, *many* horseshoe periodic orbits. We will obtain them in the next section.

4. Results: horseshoe periodic orbits for $\mu = 6.5 \cdot 10^{-9}$

Our aim in this section is to compute *stable* horseshoe periodic orbits that somewhat fit in with the real orbit described by 1980S1 and 1980S3. We will take into account that 1980S3 describes a horseshoe periodic orbit of period $\sim$ 2800 days, the minimum angular separation between both satellites is $\sim$ 6 degrees, and both satellites describe almost circular orbits of radius 1.0001452 and 0.999817 (see Spirig and Waldvogel 1985). We also want to check the mechanism of generation of horseshoe periodic orbits described above as well.

So, we consider the RTBP with $\mu = 6.5 \cdot 10^{-9}$, For this value of μ we have $C_1 = C(L_1) = 3.00001504584$, and $C_3 = C(L_3) = 3.000000013$.

In order to compute horseshoe periodic orbits, we fix a value of the Jacobi constant C and we compute them. If $C_3 < C < C_1$ then we compute the values $x_i > 0$ and $x_o > 0$ of the intersection between the zero velocity curve and the x axis. We start with a value of x very close to and bigger than x_o; from (3) we have the corresponding value of y' since we start at an orthogonal crossing, that is, the orbit begins at $(x, 0, 0, -y')$. Then, we integrate the equations of motion until the orbit crosses again the x axis. Afterwards, we increase the value of x by an increment of 10^{-7} and we go on until a change in sign of x' is found. We just compute then the initial conditions of the symmetric periodic orbit in between. Of course, we can go on increasing the value of initial x and in this way we obtain all the horseshoe periodic orbits in a given interval of x (where the separation between two consecutive values of initial conditions x is at least 10^{-7}). which enter a neighborhood of the small primary, we have regularized the differential equations of motion using Levi-Civitta variables.

It is well known that the orbit is linearly stable if the stability parameter α satisfies $-2 < \alpha < 2$ and unstable, otherwise.

In a similar way, if $3 \leq C < C_3$, the *zvc* does not cross the x axis, and if $C < 3$, then there is no *zvc* at all. In both cases, we consider as starting x a value very close to and bigger than $x(L_3)$.

We describe now the results. We have computed horseshoe periodic orbits for different values of C ranging from values less than 3 to C_1. We show, for instance, in Figure 4 the initial value x of the different horseshoe periodic orbits computed for $C = 3.000000065 \in (C_3, C_1)$. There exist thousands of horseshoe periodic orbits in the interval [1.00013179,1.0036] but as they are so close each other and for sake of clarity, we only show a particular range which includes the desired value of $r_1 = 1.0001452$. As we can see, for $x \in [1.00014, 1.00016]$ there are many horseshoe periodic orbits with suitable values of minimum angular separation (close to 6 degrees) although there are few which are stable. We plot in Figure 5 a selected *stable* horseshoe periodic orbit with $x(t = 0) = 1.00014897733$ and $x(T/2) = 0.9998510654$ and for which the minimum angular separation is approximately 6 degrees and the half period is 1372.9 days. So we can conclude that there exist stable horseshoe periodic orbits that fit with the coorbitals' motion.

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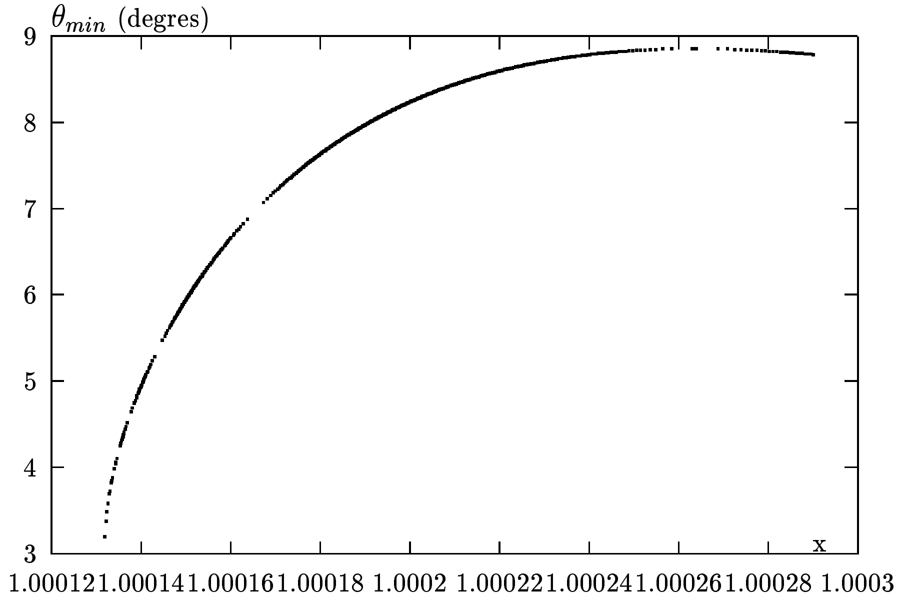

Figure 4.—Initial conditions x versus the minimum angular separation θ_{min} of horseshoe periodic orbits for $C = 3.000000065 \in (C_3, C_1)$.

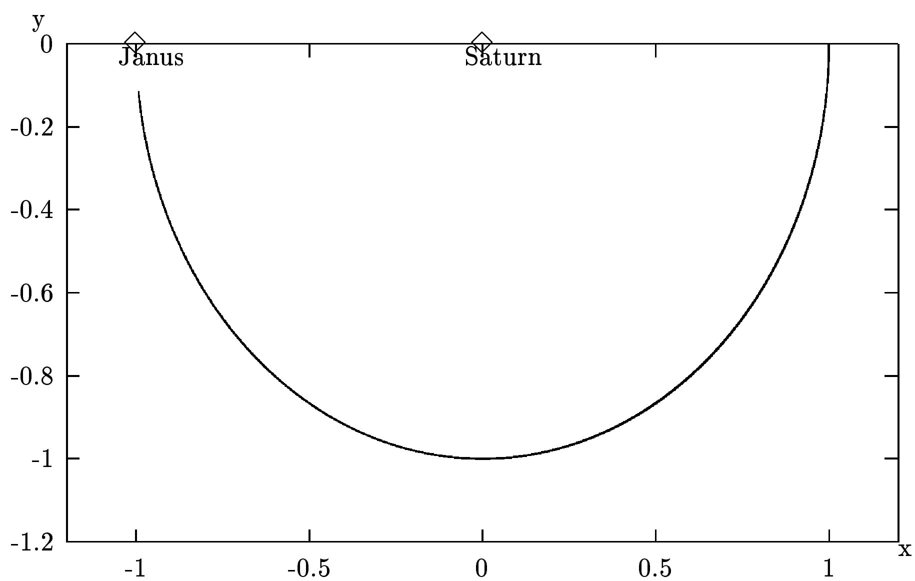

Figure 5.—A horseshoe periodic orbit suitable for Janus and Epimetheus.

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